Lec 10

Solving Linear Differential Eq.

I **Example**:

$$\ddot{y} - 3\dot{y} + 2y = 0 \quad (\star)$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

$$y(0) = 15, \quad \dot{y}(0) = 10 \quad (\star\star)$$

Find $y(t)$, which satisfies the above equation $(\star)$ and initial condition $(\star\star)$.

Soln

Write $x_1 = y$, $x_2 = \frac{dy}{dt}$

It follows that

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = 3\dot{y} - 2y = 3x_2 - 2x_1$$

10·2

We have a system of equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 3x_2 - 2x_1$$

$$x_1(0) = 15, \quad x_2(0) = 10$$

Define $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, we have

$$\dot{\mathbf{X}} = A\mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 15 \\ 10 \end{pmatrix} \qquad \star\star\star$$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

A solution of $\star\star\star$ is given by

$$\mathbf{X}(t) = e^{At}\,\mathbf{X}(0)$$

```
>> A=[0 1;-2 3]

A =

    0     1
   -2     3

>> syms t
>> X=expm(t*A)

X =

[    2*exp(t)-exp(2*t),      exp(2*t)-exp(t)]
[ -2*exp(2*t)+2*exp(t),   -exp(t)+2*exp(2*t)]

>> X0=[15;10]

X0 =

    15
    10

>> Sol=X*X0

Sol =

[   20*exp(t)-5*exp(2*t)]
[ -10*exp(2*t)+20*exp(t)]
```

$$x_1(t) = 20e^{t} - 5e^{2t} = y(t)$$

$$x_2(t) = -10e^{2t} + 20e^{t} = \dot{y}(t)$$

— x —

Eigenvalues of A are at 1 and 2.

$$e^{At} = \alpha_0 I + \alpha_1 A$$

To find α_0 and α_1 we substitute the eigenvalues and obtain

$$e^{1t} = \alpha_0 + \alpha_1 , \; e^{2t} = \alpha_0 + 2\alpha_1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}$$

Using Cramer's Rule we obtain

$$\alpha_0 = \frac{\begin{vmatrix} e^t & 1 \\ e^{2t} & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{2e^t - e^{2t}}{1}$$

$$\alpha_1 = \frac{\begin{vmatrix} 1 & e^t \\ 1 & e^{2t} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{e^{2t} - e^t}{1}$$

$$\alpha_0 = 2e^t - e^{2t}$$
$$\alpha_1 = e^{2t} - e^t$$

$$\therefore e^{At} = (2e^t - e^{2t})\underline{I} + (e^{2t} - e^t)A$$

$$e^{At}\,\mathbf{X}(0) =$$

$$(2e^{t} - e^{2t})\,\underbrace{\mathbf{X}(0)}_{= \begin{pmatrix} 15 \\ 10 \end{pmatrix}} + (e^{2t} - e^{t})\,\underbrace{A\,\mathbf{X}(0)}_{= \begin{pmatrix} 10 \\ 0 \end{pmatrix}}$$

— x —

A has eigenvector at

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for eigenvalue } \lambda = 1$$

$$v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{for eigenvalue } \lambda = 2.$$

We can write

$$\mathbf{X}(0) = \begin{pmatrix} 15 \\ 10 \end{pmatrix} = 20\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= 20\,v_1 - 5\,v_2$$

$$
\begin{aligned}
e^{At}\mathbf{x}(0) &= \\
& e^{At}(20v_1 - 5v_2) \\
&= 20\underbrace{e^{At}v_1}_{= e^{1t}v_1} - 5\underbrace{e^{At}v_2}_{= e^{2t}v_2} \\
&= 20e^{t}v_1 - 5e^{2t}v_2 \\
&= 20e^{t}\begin{pmatrix}1\\1\end{pmatrix} - 5e^{2t}\begin{pmatrix}1\\2\end{pmatrix}
\end{aligned}
$$

Solution decomposed in terms of eigenvectors.

II Example:

$$\dddot{y} + 12\ddot{y} + 47\dot{y} + 60y = 0$$

$$y(0) = 2, \quad \dot{y}(0) = 3, \quad \ddot{y}(0) = 5.$$

Find a solution $y(t)$??

~~As~~ in the previous example we define

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y},$$

it follows that

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -12x_3 - 47x_2 - 60x_1.$$

If

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad \mathbf{X}(0) = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

we have

$$\dot{\mathbf{X}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -47 & -12 \end{pmatrix} \mathbf{X}$$

10.8

$$\underline{X}(t) = e^{At}\,\underline{X}(0)$$

$$= \begin{pmatrix} 25e^{-5t} - 59e^{-4t} + 36e^{-3t} \\ -108e^{-3t} + 236e^{-4t} - 125e^{-5t} \\ -944e^{-4t} + 625e^{-5t} + 324e^{-3t} \end{pmatrix}$$

The above expression is obtained from matlab.

```
>> B=[0 1 0;0 0 1;-60 -47 -12]

B =

     0     1     0
     0     0     1
   -60   -47   -12

>> syms t
>> D=expm(t*B)

D =

[          6*exp(-5*t)-15*exp(-4*t)+10*exp(-3*t),        9/2*exp(-3*t)-8*exp(-4*t)+7/2*exp(-5*t),          1/2*exp(-3*t)-exp(-4*t)+1/2*exp(-5*t)]
[       -30*exp(-3*t)+60*exp(-4*t)-30*exp(-5*t),   -35/2*exp(-5*t)+32*exp(-4*t)-27/2*exp(-3*t),       -3/2*exp(-3*t)+4*exp(-4*t)-5/2*exp(-5*t)]
[    -240*exp(-4*t)+150*exp(-5*t)+90*exp(-3*t), 81/2*exp(-3*t)-128*exp(-4*t)+175/2*exp(-5*t),     25/2*exp(-5*t)+9/2*exp(-3*t)-16*exp(-4*t)]

>> x0=[2;3;5]

x0 =

     2
     3
     5

>> x=D*x0

x =

[      25*exp(-5*t)-59*exp(-4*t)+36*exp(-3*t)]
[  -108*exp(-3*t)+236*exp(-4*t)-125*exp(-5*t)]
[  -944*exp(-4*t)+625*exp(-5*t)+324*exp(-3*t)]
```

characteristic polynomial of A is

$$\lambda^3 + 12\lambda^2 + 47\lambda + 60 =$$
$$(\lambda+3)(\lambda+4)(\lambda+5).$$

Eigenvalues are at $-3, -4, -5$.

For the eigenvalue $\lambda = \lambda_0$, the eigenvector is at $\begin{pmatrix} 1 \\ \lambda_0 \\ \lambda_0^2 \end{pmatrix}$

$$V_1 = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix}, V_3 = \begin{pmatrix} 1 \\ -5 \\ 25 \end{pmatrix}$$

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

To find $\alpha_0, \alpha_1, \alpha_2$ we write three equations by substituting the eigenvalues and obtain the following:

$$\alpha_0 - 3\alpha_1 + 9\alpha_2 = e^{-3t}$$
$$\alpha_0 - 4\alpha_1 + 16\alpha_2 = e^{-4t}$$
$$\alpha_0 - 5\alpha_1 + 25\alpha_2 = e^{-5t}$$

$$\begin{pmatrix} 1 & -3 & 9 \\ 1 & -4 & 16 \\ 1 & -5 & 25 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ e^{-4t} \\ e^{-5t} \end{pmatrix}$$

This is the transpose of the well known Van Der Monde Matrix.

```
>> V=[1 -3 9;1 -4 16;1 -5 25]

V =
```

$$\begin{pmatrix} 1 & -3 & 9 \\ 1 & -4 & 16 \\ 1 & -5 & 25 \end{pmatrix}$$

```
>> v=det(V)

v =

    -2

>> ADJ=round(v*inv(V))

ADJ =
```

$$\begin{pmatrix} -20 & 30 & -12 \\ -9 & 16 & -7 \\ -1 & 2 & -1 \end{pmatrix}$$

adj V

```
>> inv(V)-ADJ/v

ans =
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V^{-1} = \frac{\operatorname{adj} V}{\det V}$$

$$\begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}20e^{-3t}-30e^{-4t}+12e^{-5t}\\ 9e^{-3t}-16e^{-4t}+7e^{-5t}\\ e^{-3t}-2e^{-4t}+e^{-5t}\end{pmatrix}$$

$$= \underbrace{\frac{1}{\det V}\,\mathrm{adj}\,V \begin{pmatrix}e^{-3t}\\ e^{-4t}\\ e^{-5t}\end{pmatrix}}_{\text{This is also Cramer's Rule}}$$

$$\underline{X}(t) = e^{At}\,\underline{X}(0)$$

$$= \alpha_0\,\underline{X}(0) + \alpha_1 A\,\underline{X}(0) + \alpha_2 A^2\,\underline{X}(0).$$

$$\underline{X}(0) = \begin{pmatrix}2\\3\\5\end{pmatrix},\quad A\underline{X}(0) = \begin{pmatrix}3\\5\\-321\end{pmatrix},\quad A^2\underline{X}(0) = \begin{pmatrix}5\\-321\\3437\end{pmatrix}$$

(10·12)

```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> A=[0 1 0;0 0 1;-60 -47 -12]

A =

     0     1     0
     0     0     1
   -60   -47   -12

>> X0=[2;3;5]

X0 =

     2
     3
     5

>> X1=A*X0

X1 =

     3
     5
  -321

>> X2=A*X1

X2 =

           5
        -321
        3437

>> X=[X0 X1 X2]

X =

     2             3             5
     3             5          -321
     5          -321          3437

>> syms t
>> vect=[20*exp(-3*t)-30*exp(-4*t)+12*exp(-5*t);9*exp(-3*t)-16*exp(-4*t)+7*exp(-5*t);exp(
-3*t)-2*exp(-4*t)+exp(-5*t)]/2

vect =

[ 20*exp(-3*t)-30*exp(-4*t)+12*exp(-5*t)]
[    9*exp(-3*t)-16*exp(-4*t)+7*exp(-5*t)]
[          exp(-3*t)-2*exp(-4*t)+exp(-5*t)] /2
```

$$\underline{X}(t) = \underbrace{\begin{pmatrix} \underline{X}(0) & A\underline{X}(0) & A^2\underline{X}(0) \end{pmatrix}}\underbrace{\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}}$$

```
>> Xt=X*vect/2

Xt =

[      36*exp(-3*t)-59*exp(-4*t)+25*exp(-5*t)]
[  -108*exp(-3*t)+236*exp(-4*t)-125*exp(-5*t)]
[   324*exp(-3*t)-944*exp(-4*t)+625*exp(-5*t)]
```

This answer matches with page 10·8

Finally, let us solve this problem using eigenvectors. Recall from page 10.9, that

$$v_1 = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -5 \\ 25 \end{pmatrix}$$

are the three eigenvectors corresponding to $\lambda = -3$, $\lambda = -4$, $\lambda = -5$ respectively. Writing

$$\underline{X}(0) = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = 36 \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix} + (-59) \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix} + 25 \begin{pmatrix} 1 \\ -5 \\ 25 \end{pmatrix}$$

$$\underline{X}(0) = 36\, v_1 - 59\, v_2 + 25\, v_3 .$$

$$e^{At}\underline{X}(0) = 36 \underbrace{e^{At} v_1}_{= e^{-3t} v_1} - 59 \underbrace{e^{At} v_2}_{= e^{-4t} v_2} + 25 \underbrace{e^{At} v_3}_{= e^{-5t} v_3}$$

$$e^{At}\underline{X}(0) =$$

$$e^{-3t}\,36 v_1 - e^{-4t}\,59 v_2 + e^{-5t}\,25 v_3$$

$$\underline{X}(t) = e^{At}\underline{X}(0) =$$

$$36e^{-3t}\begin{pmatrix}1\\-3\\9\end{pmatrix} - 59e^{-4t}\begin{pmatrix}1\\-4\\16\end{pmatrix} + 25e^{-5t}\begin{pmatrix}1\\-5\\25\end{pmatrix}$$

III Example:

$$\underline{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

This is the matrix on page 6.16.

$$B = \begin{pmatrix} 1782 & -990 & -396 \\ -6534 & -8514 & -4356 \\ 17,226 & 28,710 & 13,860 \end{pmatrix} \Big/ 1188$$

$$\dot{\underline{X}} = B\underline{X} \qquad \underline{X}(0) = \begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix}$$

$$\underline{X}(t) = e^{Bt}\,\underline{X}(0)$$

using matlab.

$$= \begin{pmatrix} -\frac{26}{3} t e^{2t} + 3e^{2t} \\ -\frac{286}{3} t e^{2t} + 5e^{2t} \\ \frac{754}{3} t e^{2t} + 9e^{2t} \end{pmatrix}$$

$$v_1 = \begin{pmatrix} -3 \\ -33 \\ 87 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix}$$

eigenvector of B — gen. eigenvector of B — eigenvector of B

These are on page 6·18 suitably scaled

$$B v_1 = 2 v_1, \quad B v_2 = 2 v_2 + v_1$$

$$B v_3 = 2 v_3$$

$$\boxed{\begin{aligned} e^{Bt} v_1 &= e^{2t} v_1 \\ e^{Bt} v_3 &= e^{2t} v_3 \\ e^{Bt} v_2 &= e^{2t} v_2 + t e^{2t} v_1 \end{aligned}}$$

Writing

$$\underbrace{\begin{pmatrix}3\\5\\9\end{pmatrix}}_{\mathbf{X}(0)} = \underbrace{-\frac{180}{1188}}_{\alpha}\underbrace{\begin{pmatrix}-3\\-33\\87\end{pmatrix}}_{v_1} + \underbrace{\frac{3432}{1188}}_{\beta}\underbrace{\begin{pmatrix}2\\0\\6\end{pmatrix}}_{v_2} + \underbrace{\frac{960}{1188}}_{\gamma}\underbrace{\begin{pmatrix}-4\\0\\6\end{pmatrix}}_{v_3}$$

$$\mathbf{X}(0) = \alpha v_1 + \beta v_2 + \gamma v_3$$

$$e^{Bt}\mathbf{X}(0) = \alpha e^{Bt} v_1 + \beta e^{Bt} v_2 + \gamma e^{Bt} v_3$$

$$= \alpha e^{2t} v_1 + \beta\left(e^{2t} v_2 + t e^{2t} v_1\right) + \gamma e^{2t} v_3$$

$$= (\alpha + \beta t) e^{2t} v_1 + \beta e^{2t} v_2 + \gamma e^{2t} v_3$$

Solution decomposed in terms of eigenvectors and gen. eigenvectors.

The Jordan Canonical story

If we define

$$\underset{\mathbf{X}(t)}{\underset{\shortparallel}{\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}}} = \overset{P}{\overset{\shortparallel}{\begin{pmatrix} -3 & 2 & -4 \\ -33 & 0 & 0 \\ 87 & 6 & 6 \end{pmatrix}}} \overset{Z(t)}{\overset{\shortparallel}{\begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix}}}$$

eigenvectors gen. eigenvector

of B

we obtain

$$\mathbf{X}(t) = P\,Z(t).$$

From $\dot{\mathbf{X}} = B\mathbf{X}$ it follows that

$$P\dot{Z}(t) = B\,P\,Z(t)$$

$$\Rightarrow \dot{Z}(t) = \left(P^{-1} B\, P\right) Z(t).$$

However we know that $P^{-1}BP$

is in the Jordan Canonical form.

In particular

$$P^{-1}BP = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = A$$

$$\dot{Z} = AZ$$

$$\Rightarrow \dot{z}_1 = 2z_1 + z_2$$

$$\dot{z}_2 = 2z_2$$

$$\dot{z}_3 = 2z_3$$

$$Z(0) = P^{-1}\underline{X}(0)$$

$$= \begin{pmatrix} -3 & 2 & -4 \\ -33 & 0 & 0 \\ 87 & 6 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} -180 \\ 3432 \\ 960 \end{pmatrix} \Big/ 1188$$

$$Z(t) = e^{At} Z(0)$$

The matrix A satisfies the relation

$p(A) = 0$ where $p(\lambda) = (\lambda - 2)^2$

$$= \lambda^2 - 4\lambda + 4$$

i.e $A^2 = 4A - 4I$.

Note that this is not the characteristic polynomial

Thus we have

$$e^{At} = \alpha\, I + \beta A$$

To find α, β we substitute λ for A and obtain

$$e^{\lambda t} = \alpha + \beta\lambda$$

Since λ is repeated we have

$$\frac{d}{d\lambda} e^{\lambda t} = \frac{d}{d\lambda}\left[\alpha + \beta\lambda\right]$$

Thus we have

$$\alpha + \beta\lambda = e^{\lambda t}.$$

$$\beta = t e^{\lambda t}.$$

Hence $\alpha = e^{\lambda t} - t\lambda e^{\lambda t}$

$$= (1-\lambda t) e^{\lambda t}.$$

$$\therefore e^{At} = (1-\lambda t) e^{\lambda t} I + t e^{\lambda t} A.$$

$$\Rightarrow Z(t) = e^{At} Z(0)$$

$$= (1-\lambda t) e^{\lambda t} Z(0) + t e^{\lambda t} A Z(0) \qquad \lambda = 2.$$

$$z_1(t) = -\frac{5}{33}(1-2t)e^{2t} + \frac{256}{99} t e^{2t}$$

$$z_2(t) = \frac{26}{9}(1-2t)e^{2t} + \frac{52}{9} t e^{2t}$$

$$z_3(t) = \frac{80}{99}(1-2t)e^{2t} + \frac{160}{99} t e^{2t}.$$

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$X(t) = P\,Z(t)$, hence

$$x_1(t) = 3(1-2t)e^{2t} - \frac{8}{3}te^{2t}$$

$$x_2(t) = 5(1-2t)e^{2t} - \frac{256}{3}te^{2t}$$

$$x_3(t) = 9(1-2t)e^{2t} + \frac{808}{3}te^{2t}.$$